

# PARETO OPTIMALITY AND ISOPERIMETRY

S. S. KUTATELADZE

**ABSTRACT.** Under study is the new class of geometrical extremal problems in which it is required to achieve the best result in the presence of conflicting goals; e. g., given the surface area of a convex body  $\mathfrak{x}$ , we try to maximize the volume of  $\mathfrak{x}$  and minimize the width of  $\mathfrak{x}$  simultaneously. These problems are addressed along the lines of multiple criteria decision making.

We address the multiple criteria extremal problems of convex geometry which involve the goals and constraints with the available description for the directional derivatives and the duals of the cones of feasible directions. Transition to Pareto-optimality actually involves the scalar problems with bulkier objectives. The manner of combining the geometrical and functional-analytical tools remains practically the same as in the case of a single goal typical of an isoperimetric-type problem. We proceed by way of example and present here a few model multiobjective problems that are connected with the Blaschke and Minkowski structures.

Note that we use the notation and results of [1]–[3] as regards convex geometry and the results of [4] as regards Pareto optimality.

**1. VECTOR ISOPERIMETRIC PROBLEM:** Given are some convex bodies  $\mathfrak{h}_1, \dots, \mathfrak{h}_M$ . Find a convex body  $\mathfrak{x}$  encompassing a given volume and minimizing each of the mixed volumes  $V_1(\mathfrak{x}, \mathfrak{h}_1), \dots, V_1(\mathfrak{x}, \mathfrak{h}_M)$ . In symbols,

$$\mathfrak{x} \in \mathcal{A}_N; \quad \widehat{p}(\mathfrak{x}) \geq \widehat{p}(\bar{\mathfrak{x}}); \quad (\langle \mathfrak{h}_1, \mathfrak{x} \rangle, \dots, \langle \mathfrak{h}_M, \mathfrak{x} \rangle) \rightarrow \inf.$$

Clearly, this is a Slater regular convex program in the Blaschke structure. Hence, the following holds.

**2.** *Each Pareto-optimal solution  $\bar{\mathfrak{x}}$  of the vector isoperimetric problem has the form*

$$\bar{\mathfrak{x}} = \alpha_1 \mathfrak{h}_1 + \dots + \alpha_m \mathfrak{h}_m,$$

where  $\alpha_1, \dots, \alpha_m$  are positive reals.

Let us illustrate 2 for the *Leidenfrost effect*, the spheroidal state of a liquid drop on the horizontal heated surface.

**3. LEIDENFROST PROBLEM.** Given the volume of a three-dimensional convex figure, minimize its surface area and vertical breadth.

By symmetry everything reduces to an analogous plane two-objective problem, whose every Pareto-optimal solution is by 2 a *stadium*, a weighted Minkowski sum of a disk and a horizontal straight line segment.

**4.** *A plane spheroid, a Pareto-optimal solution of the Leidenfrost problem, is the result of rotation of a stadium around the vertical axis through the center of the stadium.*

---

*Date:* February 8, 2009.

*Key words and phrases.* Isoperimetric problem, Pareto-optimum, mixed volume, Urysohn problem, Leidenfrost effect.

**5. INTERNAL URYSOHN PROBLEM WITH FLATTENING.** Given are some convex body  $\mathfrak{x}_0 \in \mathcal{V}_N$  and some flattening direction  $\bar{z} \in S_{N-1}$ . Among the convex bodies lying in  $\mathfrak{x}_0$  and having fixed integral breadth, find a convex body  $\mathfrak{x}$  trying to maximize the volume of  $\mathfrak{x}$  and minimize the breadth of  $\mathfrak{x}$  in the flattening direction:

$$\mathfrak{x} \in \mathcal{V}_N; \mathfrak{x} \subset \mathfrak{x}_0; \langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle; (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \rightarrow \inf.$$

**6.** For a feasible convex body  $\bar{\mathfrak{x}}$  to be Pareto-optimal in the internal Urysohn problem with the flattening direction  $\bar{z}$  it is necessary and sufficient that there be positive reals  $\alpha, \beta$  and a convex figure  $\mathfrak{x}$  satisfying

$$\begin{aligned} \mu(\bar{\mathfrak{x}}) &= \mu(\mathfrak{x}) + \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}}); \\ \bar{\mathfrak{x}}(z) &= \mathfrak{x}_0(z) \quad (z \in \text{supp}(\mu(\mathfrak{x})). \end{aligned}$$

By way of illustration we will derive the optimality criterion in somewhat superfluous detail. In actuality, it would suffice to appeal for instance to [4] or the other numerous sources treating Pareto optimality in slightly less generality.

Note firstly that the internal Urysohn problem with flattening may be rephrased in  $C(S_{N-1})$  as the following two-objective program

$$\begin{aligned} \mathfrak{x} &\in \mathcal{V}_N; \\ \max\{\mathfrak{x}(z) - \mathfrak{x}_0(z) \mid z \in S_{N-1}\} &\leq 0; \\ \langle \mathfrak{x}, \mathfrak{z}_N \rangle &\geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle; \\ (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) &\rightarrow \inf. \end{aligned}$$

The problem of Pareto optimization reduces to the scalar program

$$\begin{aligned} \mathfrak{x} &\in \mathcal{V}_N; \\ \max\{\max\{\mathfrak{x}(z) - \mathfrak{x}_0(z) \mid z \in S_{N-1}\}, \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle - \langle \mathfrak{x}, \mathfrak{z}_N \rangle\} &\leq 0; \\ \max\{-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})\} &\rightarrow \inf. \end{aligned}$$

The last program is Slater-regular and so we may apply the *Lagrange principle*. In other words, the value of the program under consideration coincides with the value of the unconstrained minimization problem for an appropriate Lagrangian:

$$\begin{aligned} \mathfrak{x} &\in \mathcal{V}_N; \\ \max\{-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})\} \\ + \gamma \max\{\max\{\mathfrak{x}(z) - \mathfrak{x}_0(z) \mid z \in S_{N-1}\}, \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle - \langle \mathfrak{x}, \mathfrak{z}_N \rangle\} &\rightarrow \inf. \end{aligned}$$

Here  $\gamma$  is a positive Lagrange multiplier.

We are left with differentiating the Lagrangian along the feasible directions and appealing to the available results. Note in particular that the relation  $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$  ( $z \in \text{supp}(\mu(\mathfrak{x}))$ ) is the *complementary slackness condition* standard in mathematical programming. The proof of the optimality criterion for the Urysohn problem with flattening is complete.

Assume that a plane convex figure  $\mathfrak{x}_0 \in \mathcal{V}_2$  has the symmetry axis  $A_{\bar{z}}$  with generator  $\bar{z}$ . Assume further that  $\mathfrak{x}_{00}$  is the result of rotating  $\mathfrak{x}_0$  around the symmetry axis  $A_{\bar{z}}$  in  $\mathbb{R}^3$ . In this event we come to the following problem.

**7. INTERNAL ISOPERIMETRIC PROBLEM IN THE CLASS OF THE SURFACES OF ROTATION WITH FLATTENING IN THE DIRECTION OF THE AXIS OF ROTATION:**

$$\begin{aligned} \mathfrak{x} &\in \mathcal{V}_3; \\ \mathfrak{x} &\text{ is a convex body of rotation around } A_{\bar{z}}; \\ \mathfrak{x} &\supset \mathfrak{x}_{00}; \quad \langle \mathfrak{z}_N, \mathfrak{x} \rangle \geq \langle \mathfrak{z}_N, \bar{\mathfrak{x}} \rangle; \\ &(-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \rightarrow \inf. \end{aligned}$$

By rotational symmetry, the three-dimensional problem reduces to an analogous two-dimensional problem. The integral breadth and perimeter are proportional on the plane, and we come to the already settled problem 2. Thus, we have the following.

**8.** *Each Pareto-optimal solution of 7 is the result of rotating around the symmetry axis a Pareto-optimal solution of the plane internal Urysohn problem with flattening in the direction of the axis.*

Little is known about the analogous problems in arbitrary dimensions. An especial place is occupied by the result of Porogelov who demonstrated that the “soap bubble” in a tetrahedron has the form of the result of the rolling of a ball over a solution of the internal Urysohn problem, i. e. the weighted Blaschke sum of a tetrahedron and a ball.

**9. EXTERNAL URYSOHN PROBLEM WITH FLATTENING.** Given are some convex body  $\mathfrak{x}_0 \in \mathcal{V}_N$  and some flattening direction  $\bar{z} \in S_{N-1}$ . Among the convex bodies encompassing  $\mathfrak{x}_0$  and having fixed integral breadth, find a convex body  $\mathfrak{x}$  maximizing value and minimizing breadth in the flattening direction:

$$\mathfrak{x} \in \mathcal{V}_N; \quad \mathfrak{x} \supset \mathfrak{x}_0; \quad \langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle; \quad (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \rightarrow \inf.$$

**10.** *For a feasible convex body  $\bar{\mathfrak{x}}$  to be a Pareto-optimal solution of the external Urysohn problem with flattening it is necessary and sufficient that there be positive reals  $\alpha, \beta$ , and a convex figure  $\mathfrak{x}$  satisfying*

$$\begin{aligned} \mu(\bar{\mathfrak{x}}) + \mu(\mathfrak{x}) &\gg_{\mathbb{R}^N} \alpha \mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}}); \\ V(\bar{\mathfrak{x}}) + V_1(\mathfrak{x}, \bar{\mathfrak{x}}) &= \alpha V_1(\mathfrak{z}_N, \bar{\mathfrak{x}}) + 2N\beta b_{\bar{z}}(\bar{\mathfrak{x}}); \\ \bar{\mathfrak{x}}(z) &= \mathfrak{x}_0(z) \quad (z \in \text{supp}(\mu(\mathfrak{x}))). \end{aligned}$$

Demonstration proceeds by analogy to the internal Urysohn problem with flattening. The extra equality for mixed volumes appears as the deciphering of the complementary slackness condition.

**11.** The above list may be continued with the multiobjective generalization of many scalar problems such as problems with zone constraints and current hyperplanes, problems over centrally symmetric convex figures, Lindelöf type problems, etc. These problems are usually convex with respect to Blaschke or Minkowski structures. Of greater complexity are the nonconvex parametric problems stemming from the extremal properties of the Reuleaux triangle. These problems require extra tools and undergone only a fragmentary study.

In closing we dwell upon the problems of another type where we seek for the form of several convex figures simultaneously.

**12. OPTIMAL CONVEX HULLS.** Given are convex bodies  $\eta_1, \dots, \eta_m$  in  $\mathbb{R}^N$ . Place a convex figure  $\mathfrak{x}_k$  within  $\eta_k$ , for  $k := 1, \dots, m$ , so as to simultaneously maximize the volume of each of the figures  $\mathfrak{x}_1, \dots, \mathfrak{x}_m$  and minimize the integral breadth of

the convex hull of the union of these figures:

$$\begin{aligned} \mathfrak{x}_k &\subset \mathfrak{y}_k \quad (k := 1, \dots, m); \\ (-p(\mathfrak{x}_1), \dots, -p(\mathfrak{x}_m), \langle \text{co}\{\mathfrak{x}_1, \dots, \mathfrak{x}_m\}, \mathfrak{z}_N \rangle) &\rightarrow \inf. \end{aligned}$$

**13.** For some feasible convex bodies  $\bar{\mathfrak{x}}_1, \dots, \bar{\mathfrak{x}}_m$  to have a Pareto-optimal convex hull it is necessary and sufficient that there be positive reals  $\alpha_1, \dots, \alpha_m$  not vanishing simultaneously and two collections of positive Borel measures  $\mu_1, \dots, \mu_m$  and  $\nu_1, \dots, \nu_m$  on  $S_{N-1}$  such that

$$\begin{aligned} \nu_1 + \dots + \nu_m &= \mu(\mathfrak{z}_N); \\ \bar{\mathfrak{x}}_k(z) &= \mathfrak{y}_k(z) \quad (z \in \text{supp}(\mu_k)); \\ \alpha_k \mu(\bar{\mathfrak{x}}_k) &= \mu_k + \nu_k \quad (k := 1, \dots, m). \end{aligned}$$

The criterion appears along the lines of 6.

#### REFERENCES

- [1] Kutateladze S. S. and Rubinov A. M. (1972) “Minkowski duality and its applications.” *Russian Math. Surveys*, **27**:3, 137–191.
- [2] Kutateladze S. S. and Rubinov A. M. (1976) *Minkowski Duality and Its Applications*. Novosibirsk: Nauka Publishers [in Russian].
- [3] Kutateladze S. S. (2007) “Interaction of order and convexity.” *J. Indust. Applied Math.*, **4**: 1, 399–405.
- [4] Kusraev A. G. and Kutateladze S. S. (2007) *Subdifferential Calculus: Theory and Applications*. Moscow: Nauka Publishers [in Russian].

SOBOLEV INSTITUTE OF MATHEMATICS  
4 KOPTYUG AVENUE  
NOVOSIBIRSK, 630090  
RUSSIA  
E-mail address: `sskut@member.ams.org`